

# Two graph isomorphism polytopes

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## Abstract

The convex hull  $\psi_{n,n}$  of certain  $(n!)^2$  tensors was considered recently in connection with graph isomorphism. We consider the convex hull  $\psi_n$  of the  $n!$  diagonals among these tensors. We show: 1. The polytope  $\psi_n$  is a face of  $\psi_{n,n}$ . 2. Deciding if a graph  $G$  has a subgraph isomorphic to  $H$  reduces to optimization over  $\psi_n$ . 3. Optimization over  $\psi_n$  reduces to optimization over  $\psi_{n,n}$ . In particular, this implies that the subgraph isomorphism problem reduces to optimization over  $\psi_{n,n}$ .

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## 1 Introduction

Let  $\mathcal{P}_n$  be the set of  $n \times n$  permutation matrices and consider the following two polytopes,

$$\psi_n := \text{conv}\{P \otimes P : P \in \mathcal{P}_n\}, \quad \psi_{n,n} := \text{conv}\{P \otimes Q : P, Q \in \mathcal{P}_n\}.$$

The polytope  $\psi_{n,n}$  was considered recently in [1] in connection with the graph isomorphism problem. Note that  $\psi_n$  and  $\psi_{n,n}$  have  $n!$  and  $(n!)^2$  vertices respectively. In this short note we show:

1. The polytope  $\psi_n$  is a face of the polytope  $\psi_{n,n}$ .
2. Deciding if a graph  $G$  has a subgraph isomorphic to a graph  $H$  reduces to optimization over  $\psi_n$ .
3. Optimization over  $\psi_n$  reduces to optimization over  $\psi_{n,n}$ .

In particular, this implies a result of [1] that subgraph isomorphism reduces to optimization over  $\psi_{n,n}$ .

So if  $P \neq NP$  then optimization and separation over  $\psi_n$  and hence over  $\psi_{n,n}$  cannot be done in polynomial time and a compact inequality description of  $\psi_n$  and hence of  $\psi_{n,n}$  cannot be determined.

Deciding if  $G$  has a subgraph that is isomorphic to  $H$  can also be reduced to optimization over a related polytope  $\phi_n$  defined as follows. Each permutation  $\sigma$  of the vertices of the complete graph  $K_n$  naturally induces a permutation  $\Sigma$  of its edges by  $\Sigma(\{i, j\}) := \{\sigma(i), \sigma(j)\}$ . Then  $\phi_n$  is defined as the convex hull of all  $\binom{n}{2} \times \binom{n}{2}$  permutation matrices of induced permutations  $\Sigma$ . This polytope and a broader class of so-called *Young polytopes* have been studied in [2]. In particular, therein it was shown that the graph of  $\phi_n$  is complete, so pivoting algorithms cannot be exploited for optimization over this polytope. It is an interesting question whether  $\psi_n$  and  $\phi_n$ , having  $n!$  vertices each, are isomorphic.

## 2 Statements

Define bilinear forms on  $\mathbb{R}^{n \times n}$  and on  $\mathbb{R}^{n \times n} \otimes \mathbb{R}^{n \times n}$  (note the shuffled indexation on the right) by

$$\langle A, B \rangle := \sum_{i,j} A_{i,j} B_{i,j}, \quad \langle X, Y \rangle := \sum_{i,j,s,t} X_{i,s,j,t} Y_{i,j,s,t}.$$

Let  $I$  be the  $n \times n$  identity matrix and for a graph  $G$  let  $A_G$  be its adjacency matrix. We show:

**Theorem 2.1** *The polytope  $\psi_n$  is a face of  $\psi_{n,n}$  given by*

$$\psi_n = \psi_{n,n} \cap \{X : \langle I \otimes I, X \rangle = n\}.$$

**Theorem 2.2** *Let  $G$  and  $H$  be two graphs on  $n$  vertices with  $m$  the number of edges of  $H$ . Then*

$$\max\{\langle A_G \otimes A_H, X \rangle : X \in \psi_n\} \leq 2m$$

*with equality if and only if  $G$  has a subgraph that is isomorphic to  $H$ .*

**Theorem 2.3** *Let  $W = (W_{i,s,j,t})$  be any tensor and let  $w := 2n^2 \max |W_{i,s,j,t}|$ . Then*

$$\max\{\langle W, X \rangle : X \in \psi_n\} = \max\{\langle W + wI \otimes I, X \rangle : X \in \psi_{n,n}\} - nw.$$

Combining Theorems 2.2 and 2.3 with  $W = A_G \otimes A_H$  and  $w = n^2$  (sufficing since  $W \geq 0$ , as is clear from the proof of Theorem 2.3 below), we get the following somewhat tighter form of a result of [1].

**Corollary 2.4** *Let  $G$  and  $H$  be two graphs on  $n$  vertices with  $m$  the number of edges of  $H$ . Then*

$$\max\{\langle A_G \otimes A_H + nI \otimes nI, X \rangle : X \in \psi_{n,n}\} \leq 2m + n^3$$

*with equality if and only if  $G$  has a subgraph that is isomorphic to  $H$ .*

## 3 Proofs

We record the following statement that follows directly from the definitions of the bilinear forms above.

**Proposition 3.1** *For any two simple tensors  $X = A \otimes B$  and  $Y = P \otimes Q$  we have*

$$\langle X, Y \rangle = \langle A \otimes B, P \otimes Q \rangle = \sum_{i,j,s,t} A_{i,s} B_{j,t} P_{i,j} Q_{s,t} = \langle PBQ^\top, A \rangle.$$

**Proof of Theorem 2.1.** For every  $P, Q \in \mathcal{P}_n$ , the matrix  $PIQ^\top$  is a permutation matrix, with  $PIQ^\top = I$  if and only if  $P = Q$ . It follows that for every two distinct  $P, Q \in \mathcal{P}_n$  we have

$$\langle I \otimes I, P \otimes Q \rangle = \langle PIQ^\top, I \rangle \leq n - 1 < n = \langle PIP^\top, I \rangle = \langle I \otimes I, P \otimes P \rangle. \quad \blacksquare \quad (1)$$

**Proof of Theorem 2.2.** For any  $P \in \mathcal{P}_n$ , the matrix  $PA_H P^\top$  is the adjacency matrix of the permutation of  $H$  by  $P$ . So  $\langle PA_H P^\top, A_G \rangle \leq 2m$  with equality if and only if  $H$  is isomorphic via  $P$  to a subgraph of  $G$ . Since the maximum of a linear form over a polytope is attained at a vertex we get

$$\begin{aligned} \max\{\langle A_G \otimes A_H, X \rangle : X \in \psi_n\} &= \max\{\langle A_G \otimes A_H, P \otimes P \rangle : P \in \mathcal{P}_n\} \\ &= \max\{\langle PA_H P^\top, A_G \rangle : P \in \mathcal{P}_n\} \leq 2m \end{aligned}$$

with the last inequality holding with equality if and only if  $G$  has a subgraph isomorphic to  $H$ .  $\blacksquare$

**Proof of Theorem 2.3.** For every  $P, Q \in \mathcal{P}_n$ , the tensor  $P \otimes Q = (P_{i,j} Q_{s,t})$  has  $n^2$  entries that are equal to 1 and all other entries equal to 0, and therefore  $-\frac{1}{2}w \leq \langle W, P \otimes Q \rangle \leq \frac{1}{2}w$ . Combining this with inequality (1) we see that for every two distinct  $P, Q \in \mathcal{P}_n$  we have

$$\begin{aligned} \langle W + wI \otimes I, P \otimes Q \rangle &= \langle W, P \otimes Q \rangle + w \langle I \otimes I, P \otimes Q \rangle \\ &\leq \frac{1}{2}w + (n-1)w = -\frac{1}{2}w + nw \\ &\leq \langle W, P \otimes P \rangle + w \langle I \otimes I, P \otimes P \rangle = \langle W + wI \otimes I, P \otimes P \rangle. \end{aligned}$$

Since the maximum of a linear form over a polytope is attained at a vertex we obtain the inequality

$$\begin{aligned} \max\{\langle W + wI \otimes I, X \rangle : X \in \psi_{n,n}\} &= \max\{\langle W + wI \otimes I, P \otimes Q \rangle : P, Q \in \mathcal{P}_n\} \\ &= \max\{\langle W + wI \otimes I, P \otimes P \rangle : P \in \mathcal{P}_n\} \\ &= \max\{\langle W, P \otimes P \rangle : P \in \mathcal{P}_n\} + nw \\ &= \max\{\langle W, X \rangle : X \in \psi_n\} + nw. \quad \blacksquare \end{aligned}$$

## References

- [1] S. Friedland, On the graph isomorphism problem, e-print: arXiv:0801.0398
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